

## DEGENERATIONS FOR SELF-INJECTIVE ALGEBRAS OF TREECLASS $D_n$

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### ABSTRACT

Let  $\Lambda$  be a connected representation finite selfinjective algebra. According to G. Zwara the partial orders  $\leq_{\text{ext}}$  and  $\leq_{\text{deg}}$  on the isomorphism classes of  $d$ -dimensional  $\Lambda$ -modules are equivalent if and only if the stable Auslander–Reiten quiver  $\Gamma_\Lambda$  of  $\Lambda$  is not isomorphic to  $\mathbb{Z}D_{3m}/\tau^{2m-1}$  for all  $m \geq 2$ . The paper describes all minimal degenerations  $M \leq_{\text{deg}} N$  with  $M \not\leq_{\text{ext}} N$  in the case when  $\Gamma_\Lambda \cong \mathbb{Z}D_{3m}/\tau^{2m-1}$  for some  $m \geq 2$ .

### 1. Introduction

#### 1.1. The affine variety $\text{mod}_d \Lambda$

Let  $k$  be an algebraically closed field and  $\Lambda$  be a finite dimensional associative  $k$ -algebra with unit. We denote by  $\text{mod} \Lambda$  the category of finitely generated  $\Lambda$ -left-modules. A  $d$ -dimensional  $\Lambda$ -module  $M$  is the vectorspace  $k^d$  together with a multiplication by  $\Lambda$  from the left.

Now let  $\lambda_1 = 1, \lambda_2, \dots, \lambda_n$  be a  $k$ -basis of  $\Lambda$ . Then  $\lambda_i \lambda_j = \sum_l a_{ij}^l \lambda_l$  for  $i, j = 1, \dots, n$  with the structure constants  $a_{ij}^l \in k$ . The multiplication of  $M$  by  $\lambda_i$  induces an endomorphism of  $k^d$  which we can represent by a  $d \times d$  matrix over  $k$  with respect to the standard basis of  $k^d$ . Thus  $M$  corresponds to a unique  $n$ -tuple of matrices  $m = (E, m_2, \dots, m_n) \in (\text{Mat}_{d \times d}(k))^n$ , where  $E$  denotes the identity matrix, and such an  $n$ -tuple  $m$  with  $m_1 = E$  corresponds to a  $d$ -dimensional  $\Lambda$ -module if and only if it satisfies the equations  $m_i m_j = \sum_l a_{ij}^l m_l$  for  $i, j = 1, \dots, n$ . We denote the set of all  $n$ -tuples corresponding to a  $d$ -dimensional  $\Lambda$ -module by  $\text{mod}_d \Lambda$  and we will identify the module with its  $n$ -tuple. For each  $i$  with  $1 \leq i \leq n$  let  $X^i$  denote the matrix  $(x_{\mu\nu}^i)_{\mu, \nu=1, \dots, d}$ . Then  $\text{mod}_d \Lambda$  is the zero set of the ideal  $I \subset k[x_{\mu\nu}^\xi]$  ( $\mu, \nu = 1, \dots, d; \xi = 1, \dots, n$ ), where  $I$  is generated by the components of the matrices  $X^i X^j - \sum_l a_{ij}^l X^l$  for  $i, j = 1, \dots, n$ . This gives  $\text{mod}_d \Lambda$  the structure of an affine variety, which does not have to be irreducible.

The general linear group  $\text{GL}_d(k)$  acts on  $\text{mod}_d \Lambda$  by conjugation, that is to say  $g \cdot (m_1, \dots, m_n) = (gm_1 g^{-1}, \dots, gm_n g^{-1})$  for  $g \in \text{GL}_d(k)$  and  $(m_1, \dots, m_n) \in \text{mod}_d \Lambda$ . The orbits under this action are the isomorphism classes of  $d$ -dimensional  $\Lambda$ -modules (see [7]). This definition of  $\text{mod}_d \Lambda$  depends on the chosen basis of  $\Lambda$  only up to a  $\text{GL}_d(k)$  equivariant isomorphism of affine varieties.

#### 1.2. Partial orders on isomorphism classes of $\text{mod}_d \Lambda$

A module  $N$  is called a degeneration of  $M$  (in symbols  $M \leq_{\text{deg}} N$ ) if  $N$  belongs to the Zariski closure of the  $\text{GL}_d(k)$ -orbit of  $M$  in  $\text{mod}_d \Lambda$ . Since orbits are irreducible

and open in their closure, this defines a partial order on the set of isomorphism classes of  $d$ -dimensional  $\Lambda$ -modules. It is an interesting problem to express the partial order  $\leq_{\deg}$  in algebraic terms. There are several articles in this direction, including works by S. Abeasis and A. del Fra [1], K. Bongartz [4, 5], C. Riedtmann [11] and G. Zwara [12, 14], connecting  $\leq_{\deg}$  to other partial orders on the isomorphism classes of  $d$ -dimensional  $\Lambda$ -modules.

In [15] Zwara gives an alternative description of  $\leq_{\deg}$ , that is to say  $M \leq_{\deg} N$  if and only if there exists a short exact sequence

$$0 \longrightarrow S \longrightarrow S \oplus M \longrightarrow N \longrightarrow 0 \quad (1)$$

for some  $\Lambda$ -module  $S$ .

We are concerned with two other partial orders on the isomorphism classes of  $d$ -dimensional  $\Lambda$ -modules. The partial order  $\leq_{\text{ext}}$  is the transitive closure of the relation  $M \leq_{\text{ext}} N$  if there exists a short exact sequence

$$0 \longrightarrow N_1 \longrightarrow M \longrightarrow N_2 \longrightarrow 0 \quad (2)$$

with  $N \cong N_1 \oplus N_2$ . We take the pullback of the sequence (2) with the canonical projection  $N \longrightarrow N_2$  according to the isomorphism  $N \cong N_1 \oplus N_2$ . This results in a sequence as in (1) with  $S = N_1$ , so  $\leq_{\text{ext}}$  implies  $\leq_{\deg}$ .

The hom order  $\leq$  is the partial order given by  $M \leq N$  if and only if

$$[M, X] \leq [N, X]$$

for every  $\Lambda$ -module  $X$ , where  $[U, V] := \dim_k \text{Hom}_{\Lambda}(U, V)$  for  $\Lambda$ -modules  $U$  and  $V$ . It follows immediately from (1) and the left-exactness of  $\text{Hom}_{\Lambda}(\_, X)$  that  $\leq_{\deg}$  implies  $\leq$ . The reverse implication is not true in general. However it holds for representation finite algebras (see [14]) and tame concealed algebras (see [4]).

### 1.3. Statement of the theorem

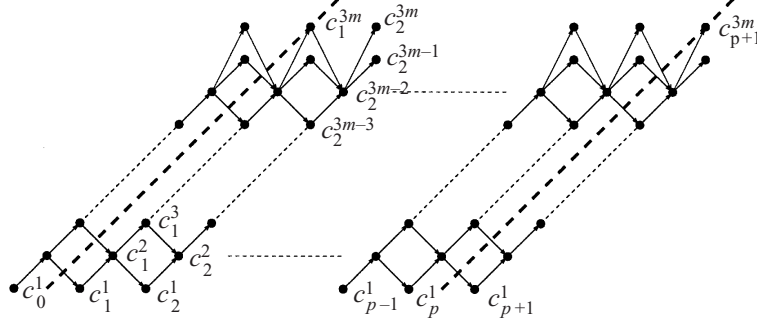
We define the Auslander–Reiten quiver  $\Gamma_{\Lambda}$  of  $\Lambda$  as the quiver whose vertices are representatives of the isomorphism classes of indecomposable  $\Lambda$ -modules. There is an arrow  $x \longrightarrow y$  between the vertices  $x$  and  $y$  if there exists an irreducible morphism from a  $\Lambda$ -module represented by  $x$  to one of  $y$ . This definition coincides with the usual one in the representation finite case (see [3]) and is appropriate for our consideration.

We denote by  $\tau$  the Auslander–Reiten translation. It is a bijection from the isomorphism classes of indecomposable non-projective  $\Lambda$ -modules to the isomorphism classes of indecomposable non-injective  $\Lambda$ -modules.

The stable Auslander–Reiten quiver  $\Gamma_{\Lambda}^s$  of  $\Lambda$  is the full subquiver of  $\Gamma_{\Lambda}$  containing all the vertices  $x$  for which  $\tau^n(x)$  is defined for all  $n \in \mathbb{Z}$ .

Let  $\Lambda$  be connected and selfinjective of finite representation type. C. Riedtmann showed in [8] that the stable Auslander–Reiten quiver  $\Gamma_{\Lambda}^s$  of  $\Lambda$  is isomorphic to  $\mathbb{Z}\Delta/G$  where  $\Delta$  is one of the Dynkin diagrams  $A_n, D_n, E_6, E_7, E_8$  and  $G$  is an admissible automorphism group of  $\mathbb{Z}\Delta$ . Using the results in [10], [9] and [6] about the category of modules over representation finite selfinjective algebras, G. Zwara showed in [13] that the partial orders  $\leq_{\text{ext}}$  and  $\leq_{\deg}$  coincide if and only if  $\Gamma_{\Lambda}^s \not\cong \mathbb{Z}D_{3m}/\tau^{2m-1}$  for all  $m \geq 2$ .

We want to investigate the difference between the partial orders  $\leq_{\deg}$  and  $\leq_{\text{ext}}$  in those exceptional cases. In particular, we want to describe the minimal degenerations

FIGURE 1.  $\mathbb{Z}D_{3m}/\tau^{2m-1}$ .

$M \leq_{\text{deg}} N$  with  $M \not\leq_{\text{ext}} N$  for a connected selfinjective algebra of finite representation type with stable Auslander–Reiten quiver  $\mathbb{Z}D_{3m}/\tau^{2m-1}$ . A degeneration  $M \leq_{\text{deg}} N$  is called minimal if it is a proper degeneration, that is to say  $M \not\cong N$ , and if there exists no module  $P$  with  $M \not\cong P \not\cong N$  and  $M \leq_{\text{deg}} P \leq_{\text{deg}} N$ . It is an interesting question how complicated minimal degenerations are. Some results concerning the complexity of degenerations can be found in [2].

G. Zwara proved in [15, Theorem 4] that for a minimal degeneration  $M \leq_{\text{deg}} N$  with  $M \not\leq_{\text{ext}} N$  there exist decompositions  $M \cong M' \oplus W$  and  $N \cong N' \oplus W$  such that  $N'$  is indecomposable and  $M' \leq_{\text{deg}} N'$  is a minimal degeneration. Therefore it is enough to concentrate on degenerations to indecomposables.

The stable translation quiver  $\mathbb{Z}D_{3m}$  has the vertices  $c_j^i$  where  $i \in \{1, \dots, 3m\}$  and  $j \in \mathbb{Z}$ . There are arrows  $c_j^i \rightarrow c_{j+1}^i$  and  $c_{j+1}^i \rightarrow c_j^{i+1}$  for  $1 \leq i \leq 3m-2$  and arrows  $c_j^{3m-2} \rightarrow c_j^{3m}$  and  $c_j^{3m} \rightarrow c_{j+1}^{3m-2}$ . The translation is given by  $\tau(c_j^i) = c_{j-1}^i$ . Thus the vertices  $c_j^i$  and  $c_{j+2m-1}^i$  are identified in the quotient  $\mathbb{Z}D_{3m}/\tau^{2m-1}$ . In Figure 1 the stable Auslander–Reiten quiver  $\mathbb{Z}D_{3m}/\tau^{2m-1}$  is drawn for  $m$  even. Every letter refers to the vertex at its left and the thick diagonal lines indicate the  $(2m-1)$ -period of the translation  $\tau$ .

**THEOREM 1.1.** *Let  $\Lambda$  be a connected and selfinjective algebra of finite representation type whose stable Auslander–Reiten quiver is isomorphic to  $\mathbb{Z}D_{3m}/\tau^{2m-1}$ . There exists a proper degeneration  $M \leq_{\text{deg}} N$  to the indecomposable  $\Lambda$ -module  $N$  if and only if  $N$  corresponds to a vertex  $c_l^s$  with  $m+1 \leq s \leq 2m$ . Moreover, the module  $M$  is determined by  $N$  up to isomorphism.*

## 2. Preliminaries

We want to represent pairs of modules  $(M, N)$  in terms of  $\mathbb{Z}$ -valued difference-functions on the set of isomorphism classes of indecomposable modules and to characterize those functions corresponding to pairs  $(M, N)$  with  $M \leq_{\text{deg}} N$ . This will enable us to give a combinatorial proof of the theorem.

We say that the modules  $M$  and  $N$  are disjoint if they have no common direct summand. We denote by  $\bar{M}$  the isomorphism class of the module  $M$  and by  $\mathcal{S}$  the set of ordered pairs  $(\bar{M}, \bar{N})$  such that  $M$  and  $N$  are disjoint. To every pair  $(\bar{M}, \bar{N})$  we associate the function  $\delta_{M,N}$  given by  $\delta_{M,N}(X) = [N, X] - [M, X]$ .

Let  $\mu(X, A)$  be the multiplicity of the indecomposable direct summand  $X$  in the direct sum decomposition of  $A$ . In particular  $A \cong \bigoplus_{\bar{X}: X \text{ indec}} X^{\mu(X,A)}$ .

$$\begin{array}{ccc}
& \oplus_{\bar{X}; X \text{ indec.}} \mathbb{Z} \bar{X} & \\
\alpha \nearrow & & \searrow \beta \\
\mathcal{S} & \xrightarrow{\gamma} & (\oplus_{\bar{Y}; Y \text{ indec.}} \mathbb{Z} \bar{Y})^*
\end{array}$$

FIGURE 2.

Let  $(\oplus_{\bar{Y}; Y \text{ indec.}} \mathbb{Z} \bar{Y})^* = \text{Hom}_{\mathbb{Z}}(\oplus_{\bar{Y}; Y \text{ indec.}} \mathbb{Z} \bar{Y}, \mathbb{Z})$ . We consider the diagram in Figure 2 where  $\alpha, \beta$  and  $\gamma$  are given by

$$\begin{aligned}
\alpha(\bar{M}, \bar{N}) &= \sum_{\bar{X}; X \text{ indec.}} (\mu(X, N) - \mu(X, M)) \bar{X}, \\
\gamma(\bar{M}, \bar{N}) &= \delta_{M, N},
\end{aligned}$$

$$\beta \left( \sum_{\bar{X}; X \text{ indec.}} \lambda_{\bar{X}} \bar{X} \right) = \sum_{\bar{X}; X \text{ indec.}} \lambda_{\bar{X}} [X, \quad ].$$

The diagram commutes since

$$\beta \circ \alpha(\bar{M}, \bar{N}) = \sum_{\bar{X}} (\mu(X, N) - \mu(X, M)) [X, \quad ] = [N, \quad ] - [M, \quad ] = \delta_{M, N}.$$

Obviously  $\alpha$  is a bijection and  $\beta$  is  $\mathbb{Z}$ -linear.

LEMMA 2.1. *If  $\Lambda$  is of finite representation type then  $\beta$  is an isomorphism.*

*Proof.* The map  $\beta$  is  $\mathbb{Z}$ -linear between free  $\mathbb{Z}$ -modules of the same finite rank. Thus it suffices to show that  $\beta$  is surjective. For each indecomposable module  $X$  we consider the exact sequence

$$X \longrightarrow E'_X \longrightarrow \tau^{-1}X \longrightarrow 0$$

which is the Auslander–Reiten sequence starting in  $X$  if  $X$  is not injective. Otherwise we set  $E'_X = X/\text{soc}(X)$  and  $\tau^{-1}X = 0$ . The functor  $\text{Hom}_{\Lambda}(\quad, Y)$  induces the exact sequence

$$0 \longrightarrow \text{Hom}_{\Lambda}(\tau^{-1}X, Y) \longrightarrow \text{Hom}_{\Lambda}(E'_X, Y) \longrightarrow \text{Hom}_{\Lambda}(X, Y) \longrightarrow k^{\mu(X, Y)} \longrightarrow 0$$

of  $k$ -vectorspaces. Thus for every indecomposable module  $Y$  we have

$$([X, \quad ] + [\tau^{-1}X, \quad ] - [E'_X, \quad ])(Y) = \begin{cases} 1 & \text{if } Y \cong X \\ 0 & \text{otherwise,} \end{cases}$$

showing that  $\beta$  is surjective. □

We want to describe the inverse of  $\beta$ . For each indecomposable module  $X$  we consider the exact sequence

$$0 \longrightarrow \tau X \longrightarrow E_X \longrightarrow X$$

which is the Auslander–Reiten sequence ending in  $X$  if  $X$  is not projective. Otherwise we set  $E_X = \text{rad}(X)$  and  $\tau X = 0$ . Then  $\beta^{-1}$  is given by

$$\beta^{-1}(\delta) = \sum_{\bar{X}; X \text{ indec.}} (\delta(X) + \delta(\tau X) - \delta(E_X)) \bar{X}.$$

Thus, if  $\Lambda$  is of finite representation type,  $\gamma$  is bijective and we write  $\gamma^{-1}(\delta) = (\bar{M}_\delta, \bar{N}_\delta) \in \mathcal{S}$ . Then

$$\begin{aligned} \sum_{\bar{X}; X \text{ indec}} (\delta(X) + \delta(\tau X) - \delta(E_X)) \bar{X} &= \beta^{-1}(\delta) = \alpha \circ \gamma^{-1}(\delta) \\ &= \sum_{\bar{X}; X \text{ indec}} (\mu(X, N_\delta) - \mu(X, M_\delta)) \bar{X} \end{aligned}$$

and in consequence

$$\delta(X) + \delta(\tau X) - \delta(E_X) = \mu(X, N_\delta) - \mu(X, M_\delta) \quad (3)$$

for every indecomposable module  $X$  and every  $\delta \in (\bigoplus_{\bar{Y}; Y \text{ indec}} \mathbb{Z}\bar{Y})^*$ .

Let  $\mathcal{S}' \subset \mathcal{S}$  be the subset containing all pairs  $(\bar{M}, \bar{N})$  with  $M \leq_{\deg} N$ .

**LEMMA 2.2.** *If  $\Lambda$  is of finite representation type then  $\gamma$  restricts to a bijection between  $\mathcal{S}'$  and the set of non-negative functions  $\delta \in (\bigoplus_{\bar{Y}; Y \text{ indec}} \mathbb{Z}\bar{Y})^*$  such that  $\delta(I) = 0$  for every injective module  $I$ .*

*Proof.* Zwara showed in [14] that the partial orders  $\leq_{\deg}$  and  $\leq$  coincide for representation finite algebras. Hence  $\gamma(\bar{M}, \bar{N})$  is a non-negative function for every  $(\bar{M}, \bar{N}) \in \mathcal{S}'$ . If  $I$  is an injective module then  $[N, I] = [M, I]$  holds in consequence of the exactness of  $\text{Hom}_\Lambda(-, I)$  and (1). On the other hand, let  $\delta \in (\bigoplus_{\bar{Y}; Y \text{ indec}} \mathbb{Z}\bar{Y})^*$  be non-negative such that  $\delta(I) = 0$  for every injective module. We have to show that  $\dim_k N_\delta = \dim_k M_\delta$  holds. We consider the injective module  $\text{Hom}_k(\Lambda_\Lambda, k)$ , where  $\Lambda_\Lambda$  denotes  $\Lambda$  as  $\Lambda$ -right module. Then the adjoint isomorphism gives  $[A, \text{Hom}_k(\Lambda_\Lambda, k)] = \dim_k \text{Hom}_k(A, k) = \dim_k A$  for every  $\Lambda$ -module  $A$ . In particular  $\dim_k N_\delta = \dim_k M_\delta$ .  $\square$

Let  $\mathcal{S}'_N = \{(\bar{X}, \bar{Y}) \in \mathcal{S}' \mid \bar{Y} = \bar{N}\}$ . As a consequence of Lemma 2.2 and (3) we can describe all isomorphism classes of modules degenerating to an indecomposable.

**LEMMA 2.3.** *Let  $\Lambda$  be of finite representation type and  $N$  be indecomposable. Then  $\mathcal{S}'_N$  is mapped bijectively by  $\gamma$  to the set of non-negative functions  $\delta \in (\bigoplus_{\bar{Y}; Y \text{ indec}} \mathbb{Z}\bar{Y})^*$  with  $\delta(I) = 0$  for every injective module  $I$ , and satisfying*

$$\delta(X) + \delta(\tau X) - \delta(E_X) \begin{cases} = 1 & \text{if } X \cong N \\ \leq 0 & \text{otherwise} \end{cases} \quad (4)$$

for every indecomposable  $X$ .

Note that for  $(\bar{M}, \bar{N}) = \gamma^{-1}(\delta)$  we have then

$$M \cong \bigoplus_{\bar{X}; X \not\cong N} X^{-(\delta(X) + \delta(\tau X) - \delta(E_X))}, \quad (5)$$

where the direct sum is taken over all isomorphism classes of indecomposable  $\Lambda$ -modules except that of  $N$ .

### 3. Proof of Theorem 1.1

Let  $\Lambda$  be a selfinjective finite dimensional  $k$ -algebra of finite representation type with stable Auslander–Reiten quiver isomorphic to  $\mathbb{Z}D_{3m}/\tau^{2m-1}$ . If  $M \leq_{\deg} N$  is a

proper degeneration to the indecomposable  $N$ , then  $M$  and  $N$  are disjoint. If  $N$  were projective then the sequence (1) would split in contradiction to  $M \not\cong N$ . Thus  $N$  corresponds to a vertex  $c_l^s$  in the stable Auslander–Reiten quiver of  $\Lambda$ .

In subsection 3.1 we will characterize a function  $\delta \in \gamma(\mathcal{S}'_N)$  describing a proper degeneration to  $N$  as the unique solution of a linear system depending on two vertices. One of these vertices is the vertex corresponding to  $N$ . In Subsection 3.2 we analyse this linear system. We will show that if this linear system has a solution in the natural numbers, then this solution is uniquely determined by the vertex  $c_l^s$  and  $m+1 \leq s \leq 2m$  holds. It follows then from Lemma 2.3 that there exists up to isomorphism at most one module  $M$  degenerating to  $N$ . Finally we give, for the indecomposable module  $N$  corresponding to the vertex  $c_l^s$  with  $m+1 \leq s \leq 2m$ , a non-negative function  $\delta$  which satisfies (4) for every indecomposable. Thus by Lemma 2.3 there exists a proper degeneration  $M \leq_{\text{deg}} N$ .

By reindexing the stable Auslander–Reiten quiver  $\mathbb{Z}D_{3m}/\tau^{2m-1}$  we can assume that  $l = 1$ . From now on  $N$  always corresponds to the vertex  $c_1^s$ .

### 3.1. Characterization of $\delta$ by a linear system

We denote by  $p := 2m - 1$  the period of the Auslander–Reiten translation  $\tau$  and by  $h := 3m - 1$  the ‘height’ of the Dynkin diagram  $D_{3m}$ .

Let us fix an element  $\delta \in \gamma(\mathcal{S}'_N)$ . We set

$$a_j^i := \begin{cases} 0 & \text{if } i = 0 \\ \delta(c_j^i) & \text{if } 1 \leq i \leq h-1 \\ b_j^h + b_j^{h+1} & \text{if } i = h, \end{cases}$$

where  $b_j^i := \delta(c_j^i)$  for  $h \leq i \leq h+1$ .

Note that all the integers  $a_j^i$  and  $b_j^i$  are non-negative. We consider for each vertex  $c_j^i$  the Auslander–Reiten sequence ending in  $c_j^i$ . Since  $\delta(I) = 0$  if  $I$  is an injective  $\Lambda$ -module we obtain the following set of inequalities from Lemma 2.3.

If  $1 \leq i \leq h-1$

$$a_j^i + a_{j-1}^i - a_{j-1}^{i+1} - a_j^{i-1} \begin{cases} = 1 & \text{if } c_j^i = c_1^s \\ \leq 0 & \text{otherwise.} \end{cases} \quad (6)$$

If  $h \leq i \leq h+1$

$$b_j^i + b_{j-1}^i - a_j^{h-1} \begin{cases} = 1 & \text{if } c_j^i = c_1^s \\ \leq 0 & \text{otherwise.} \end{cases} \quad (7)$$

These inequalities are the key to proving the theorem. First we derive some information on the  $\tau$ -orbits in  $\mathbb{Z}D_{3m}/\tau^{2m-1}$ . We sum up the  $\delta$ -values along each  $\tau$ -orbit and set

$$a_i = \sum_{j=1}^p a_j^i \quad \text{for } 0 \leq i \leq h,$$

$$b_i = \sum_{j=1}^p b_j^i \quad \text{for } h \leq i \leq h+1.$$

Then by definition  $a_h = b_h + b_{h+1}$  and  $a_0 = 0$ . For each fixed  $i$  we add up the inequalities of (6) and (7) respectively and we obtain

$$2a_i \leq a_{i+1} + a_{i-1} + \delta_{i,s} \quad \text{for } 1 \leq i \leq h-1, \quad (8)$$

$$2b_i \leq a_{h-1} + \delta_{i,s} \quad \text{for } h \leq i \leq h+1. \quad (9)$$

Here  $\delta_{i,s}$  denotes the Kronecker symbol. By definition and the inequality (9) we get  $2a_h = 2b_h + 2b_{h+1} \leq 2a_{h-1} + 1$ . Hence

$$a_h \leq a_{h-1}. \quad (10)$$

REMARK 3.1. From  $\delta(N) + \delta(\tau N) - \delta(E_N) = 1$  it follows immediately that  $a_s \geq \delta(N) + \delta(\tau N) > 0$  if  $s \leq h-1$  and  $a_h > 0$  if  $s \geq h$ .

The following lemma implies that the case  $s \geq h$  does not occur. In view of Figure 1 this means that the vertex  $c_1^s$  is not one of the somehow exceptional vertices on the upper boundary.

LEMMA 3.2. *It holds that  $s \leq h-1$  and there exists an integer  $t$  with  $2 \leq t \leq s$  such that*

$$t = 2b_h = 2b_{h+1} \quad \text{and} \quad a_i = \begin{cases} 0 & \text{if } 0 \leq i \leq s-t \\ i - (s-t) & \text{if } s-t \leq i \leq s \\ t & \text{if } s \leq i \leq h. \end{cases}$$

*In particular  $a_s = t$  is an even integer.*

*Proof.* The inequalities in (8) are equivalent to

$$a_i - a_{i-1} \leq a_{i+1} - a_i + \delta_{i,s} \quad (11)$$

for  $1 \leq i \leq h-1$ . Suppose that  $s \geq h$ . It follows from (11) and (10) that

$$0 \leq a_1 = a_1 - a_0 \leq \dots \leq a_h - a_{h-1} \leq 0.$$

This implies that  $a_i = 0$  for all  $i \in \{1, \dots, h\}$  in contradiction to  $a_h > 0$  by Remark 3.1. Hence  $s \leq h-1$ .

Again by (11) and (10) we obtain the following chain of inequalities:

$$\begin{aligned} 0 \leq a_1 = a_1 - a_0 &\leq a_2 - a_1 \leq \dots \leq a_s - a_{s-1} \\ &\leq a_{s+1} - a_s + 1 \leq \dots \leq a_h - a_{h-1} + 1 \leq 1. \end{aligned}$$

If  $a_s - a_{s-1} = 0$  then  $a_1 = a_2 = \dots = a_s = 0$  in contradiction to  $a_s > 0$  by Remark 3.1. Hence there is an integer  $t$  with  $0 < t \leq s$  such that

$$\begin{aligned} 0 &= a_1 - a_0 = \dots = a_{s-t} - a_{s-t-1}, \\ 1 &= a_{s-t+1} - a_{s-t} = \dots = a_s - a_{s-1} \\ &= a_{s+1} - a_s + 1 = \dots = a_h - a_{h-1} + 1. \end{aligned}$$

Our claim for the  $a_i$  is an easy consequence. In particular we have  $a_{h-1} = a_h = b_h + b_{h+1}$ , but  $2b_{h+1} \leq a_{h-1}$  and  $2b_h \leq a_{h-1}$  by (9). Hence we see that  $t = a_{h-1} = 2b_h = 2b_{h+1}$  is an even integer.  $\square$

As an immediate consequence of Lemma 3.2 we note that

$$2a_i - a_{i+1} - a_{i-1} = \begin{cases} 1 & \text{if } i = s \\ -1 & \text{if } i = s - t \\ 0 & \text{otherwise} \end{cases} \quad (12)$$

for  $1 \leq i \leq h - 1$ .

We want to describe the non-negative integers  $a_j^i$  and  $b_j^i$  and hence the function  $\delta$  as the unique solution of a linear system. By Lemma 3.2 we have  $1 = a_{s-t+1} = \sum_j \delta(c_j^{s-t+1})$ , so there exists exactly one vertex  $c_{u-1}^{s-t+1}$  with  $\delta(c_{u-1}^{s-t+1}) = a_{u-1}^{s-t+1} = 1$ . Note that the index  $u$  is only determined modulo  $p$ . In the sequel let  $\tilde{u}$  be the representative of  $u$  with  $1 \leq \tilde{u} \leq p$ .

If  $s > t$  the Auslander–Reiten sequence ending in  $c_u^{s-t}$  gives rise to the equation

$$a_u^{s-t} + a_{u-1}^{s-t} - a_{u-1}^{s-t+1} - a_u^{s-t-1} = -1 \quad (13)$$

because  $a_i = 0$  for  $i \leq s - t$ , by Lemma 3.2.

We consider the following linear system which depends on the positions of the vertices  $c_1^s$  and  $c_{u-1}^{s-t+1}$  or equivalently on the integers  $s, t$  and  $\tilde{u}$ . The lower index is taken to be in  $\mathbb{Z}/p\mathbb{Z}$ .

$$x_j^0 = 0, \quad x_j^1 = \begin{cases} 1 & \text{if } c_j^1 = c_{u-1}^{s-t+1} \\ 0 & \text{otherwise,} \end{cases} \quad x_j^h = y_j^h + y_j^{h+1}. \quad (14)$$

If  $1 \leq i \leq h - 1$

$$x_j^i + x_{j-1}^i - x_{j-1}^{i+1} - x_j^{i-1} = \begin{cases} 1 & \text{if } c_j^i = c_1^s \\ -1 & \text{if } c_j^i = c_u^{s-t} \\ 0 & \text{otherwise.} \end{cases} \quad (15)$$

If  $h \leq i \leq h + 1$

$$y_j^i + y_{j-1}^i = x_j^{h-1}. \quad (16)$$

**LEMMA 3.3.** *If  $x_j^i, y_j^i$  is a rational solution of the linear system (14)–(16) then  $y_j^{h+1} = y_j^h$  holds for all  $j$ .*

*Proof.* Suppose that there is  $j_0$  with  $y_{j_0}^{h+1} > y_{j_0}^h$ . Since  $y_j^{h+1} + y_{j+1}^{h+1} = x_{j+1}^{h-1} = y_j^h + y_{j+1}^h$  by (16) we have  $-y_{j_0+1}^{h+1} > -y_{j_0+1}^h$  and successively

$$-y_{j_0}^{h+1} = (-1)^p y_{j_0+p}^{h+1} > (-1)^p y_{j_0+p}^h = -y_{j_0}^h, \quad \text{as } p \text{ is odd,}$$

in contradiction to  $y_{j_0}^{h+1} > y_{j_0}^h$ .  $\square$

To any integer solution  $x_j^i, y_j^i$  of this linear system we can associate a function  $\delta' \in (\bigoplus_{\bar{Y}; Y \text{ indec}} \mathbb{Z}\bar{Y})^*$  by setting  $\delta'(c_j^i) = x_j^i$  for  $1 \leq i \leq h - 1$ ,  $\delta'(c_j^i) = y_j^i$  for  $h \leq i \leq h + 1$  and  $\delta'(I) = 0$  for every injective module  $I$ . Under the same conditions we will speak of a function  $\delta' \in (\bigoplus_{\bar{Y}; Y \text{ indec}} \mathbb{Z}\bar{Y})^*$  as a solution of the linear system (14)–(16).



LEMMA 3.4. *The unique solution of the linear system (14)–(16) is given by  $x_j^i = a_j^i$  and  $y_j^i = b_j^i$ . In particular  $a_j^h = 2b_j^h$  is an even integer for all  $j$ .*

*Proof.* With respect to uniqueness, it is obvious that the values of  $x_j^i$  are determined by the equations in (14) and (15). The values  $y_j^i$  are given by  $y_j^i = x_j^h/2$  by (14) and Lemma 3.3.

The equations in (14) are obviously satisfied by  $a_j^i$  and  $b_j^i$ .

Because of (13) and  $a_1^s + a_0^s - a_0^{s+1} - a_1^{s-1} = 1$  it remains for (15) to show that

$$a_j^i + a_{j-1}^i - a_{j-1}^{i+1} - a_j^{i-1} = 0$$

for  $c_u^{s-t} \neq c_j^i \neq c_1^s$ . By (12) we have

$$0 = 2a_i - a_{i+1} - a_{i-1} = \sum_j (a_j^i + a_{j-1}^i - a_{j-1}^{i+1} - a_j^{i-1})$$

for  $s \neq i \neq s-t$  and

$$0 = 2a_{s-t} - a_{s-t+1} - a_{s-t-1} + 1 = \sum_{j \neq u} (a_j^{s-t} + a_{j-1}^{s-t} - a_{j-1}^{s-t+1} - a_j^{s-t-1})$$

for  $i = s-t$  and

$$0 = 2a_s - a_{s+1} - a_{s-1} - 1 = \sum_{j \neq 1} (a_j^s + a_{j-1}^s - a_{j-1}^{s+1} - a_j^{s-1})$$

for  $i = s$ . However because of (6) each of the summands on the right-hand side is less than or equal to zero. Thus each summand on the right-hand side is zero.

Concerning (16) we remark that for  $h \leq i \leq h+1$  we have

$$0 = 2b_i - a_{h-1} = \sum_j (b_j^i + b_{j-2}^i - a_{j-1}^{h-1})$$

by Lemma 3.2. Since each summand on the right-hand side is less than or equal to zero by (7) it is zero.

Finally we have  $a_j^h = 2b_j^h$  by (14) and Lemma 3.3.  $\square$

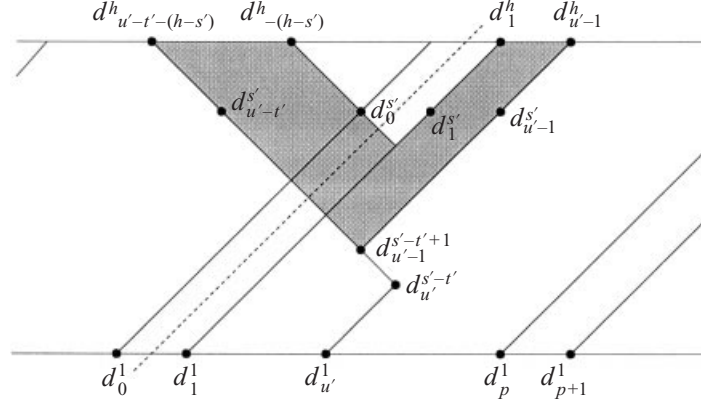
This means that every  $\delta \in \gamma(\mathcal{S}'_N)$  is the unique solution of a linear system which itself depends on some vertex  $c_{u-1}^{s-t+1}$ . In the next section we will show that  $c_{u-1}^{s-t+1}$  is uniquely determined by  $c_1^s$ .

### 3.2. Computation of $c_{u-1}^{s-t+1}$

Note that the vertices  $c_1^s$  and  $c_{u-1}^{s-t+1}$  have to be positioned in  $\mathbb{Z}D_{3m}/\tau^{2m-1}$  in such a way that the unique solution of the linear system (14)–(16) takes values in  $\mathbb{N}$ . We will show that this condition implies that  $m+1 \leq s \leq 2m$  and determines  $c_{u-1}^{s-t+1}$  as  $c_{s-m}^{2m-s+1}$ .

It is possible to solve this problem directly by examining the linear system (14)–(16), but this procedure is rather complicated. Therefore we use a different method. We will ignore the equations of (16) and only use that  $a_h^j$  is even. Furthermore we use a covering technique to simplify the computations.

To this purpose we consider the stable translation-quiver  $\mathbb{Z}A_h$  with vertices  $d_j^i$  where  $1 \leq i \leq h$  and  $j \in \mathbb{Z}$ . There are arrows from  $d_j^i$  to  $d_j^{i+1}$  and from  $d_j^{i+1}$  to  $d_{j+1}^i$  for  $1 \leq i \leq h-1$ . The translation  $\tau$  is given by  $\tau(d_j^i) = d_{j-1}^i$ . In particular we are interested in two quotients of  $\mathbb{Z}A_h$ , namely  $Q_1 := \mathbb{Z}A_h/\tau^p$  and  $Q_2 := \mathbb{Z}A_h/\tau^{2p}$ .

FIGURE 3.  $Q_2$ .

Now let  $s', t', u' \in \mathbb{N}$  with  $2 \leq t' \leq s' \leq h-1$  and  $1 \leq u' \leq p$ . Let  $\delta_r : \{\text{vertices of } Q_r\} \rightarrow \mathbb{Z}$  for  $r = 1, 2$  be a function satisfying

$$\delta_r(d_j^1) = \begin{cases} 1 & \text{if } d_j^1 = d_{u'-1}^{s'-t'+1} \\ 0 & \text{otherwise,} \end{cases} \quad (17)$$

$$\delta_r(d_j^1) + \delta_r(d_{j-1}^1) - \delta_r(d_{j-1}^2) = \begin{cases} -1 & \text{if } d_j^1 = d_{u'}^{s'-t'} \\ 0 & \text{otherwise,} \end{cases} \quad (18)$$

$$\delta_r(d_j^i) + \delta_r(d_{j-1}^i) - \delta_r(d_{j-1}^{i+1}) - \delta_r(d_j^{i-1}) = \begin{cases} 1 & \text{if } d_j^i = d_1^{s'} \\ -1 & \text{if } d_j^i = d_{u'}^{s'-t'} \\ 0 & \text{otherwise,} \end{cases} \quad (19)$$

for  $2 \leq i \leq h-1$ .

The values of  $\delta_r(d_j^1)$  are determined by condition (17). The values of  $\delta_r(d_j^i)$  with  $i \geq 2$  are determined by the values of  $\delta_r(d_j^{i'})$  with  $i' < i$  because of conditions (18) and (19). Thus there exists exactly one such function  $\delta_r$ .

The function  $\delta_2$  is easy to calculate.

**LEMMA 3.5.** *If  $u' - t' \leq 1$  then the function  $\delta_2$  is given by  $\delta_2(d_j^i) = 1$  if the vertex  $d_j^i$  lies in the shaded area (including the boundary) of Figure 3 and  $\delta_2(d_j^i) = 0$  otherwise.*

*Proof.* It is straightforward to check that Equations (17), (18) and (19) hold.  $\square$

We define the function  $\delta'_1 : \{\text{vertices of } Q_1\} \rightarrow \mathbb{Z}$  by  $\delta'_1(d_j^i) = \delta_2(d_j^i) + \delta_2(d_{j+p}^i)$ . Of course  $\delta'_1$  satisfies Equations (17), (18) and (19), from which we see that  $\delta'_1 = \delta_1$  and consequently

$$\delta_1(d_j^i) = \delta_2(d_j^i) + \delta_2(d_{j+p}^i). \quad (20)$$

We note the following.

**LEMMA 3.6.** *If  $u' - t' \leq 1$  then  $\delta_1(d_j^i)$  is an even integer if and only if  $\delta_2(d_j^i) = \delta_2(d_{j+p}^i)$ .*

*Proof.* Since  $\delta_2$  takes only values in  $\{0, 1\}$  by Lemma 3.5 the claim follows from (20).  $\square$

Let us now consider again our function  $\delta \in \gamma(\mathcal{S}'_N)$ . This function induces a function  $\bar{\delta} : \{\text{vertices of } Q_1\} \rightarrow \mathbb{Z}$  by  $\bar{\delta}(d_j^i) = a_j^i$ . We set  $s' = s$ ,  $t' = t$  and  $u' = \tilde{u}$ .

Then the function  $\bar{\delta}$  satisfies Equations (17), (18) and (19) by Lemma 3.4 and consequently  $\bar{\delta} = \delta_1$ .

LEMMA 3.7. *If  $\delta \in \gamma(\mathcal{S}'_N)$  is the unique solution of the linear system (14)–(16) then we have  $\tilde{u} - t \leq 1$ .*

*Proof.* Suppose the lemma is false. Then  $1 < \tilde{u} - t \leq \tilde{u} \leq p$  and in consequence none of the integers  $2p, 1, p, p+1$  is congruent modulo  $2p$  to an integer in  $\{\tilde{u} - t, \tilde{u} - t + 1, \dots, \tilde{u} - 1\}$ . By (17)–(19) and since  $2 \leq t < p$  we have

$$\delta_2(d_j^s) = \begin{cases} 1 & \text{if } d_j^s = d_{j'}^s \text{ with } j' \in \{\tilde{u} - t, \tilde{u} - t + 1, \dots, \tilde{u} - 1\} \\ 0 & \text{otherwise.} \end{cases}$$

Hence by (20) and Remark 3.1

$$\begin{aligned} 0 &= \delta_2(d_1^s) + \delta_2(d_{1+p}^s) + \delta_2(d_{2p}^s) + \delta_2(d_p^s) \\ &= \bar{\delta}(d_1^s) + \bar{\delta}(d_p^s) \\ &= \delta(N) + \delta(\tau N) \\ &\geq 1 \end{aligned}$$

which is obviously a contradiction.  $\square$

We are able now to determine  $\tilde{u}$  and  $t$  by means of  $s$ .

LEMMA 3.8. *If  $\delta \in \gamma(\mathcal{S}'_N)$  solves the linear system (14)–(16) then  $c_{u-1}^{s-t+1} = c_{s-m}^{2m-s+1}$  and  $m+1 \leq s \leq 2m$  holds. In particular  $t = 2(s-m)$  and  $\tilde{u} = t/2 + 1$ .*

*Proof.* By Lemma 3.7 we have  $\tilde{u} - t \leq 1$ . Therefore we can apply Lemma 3.5 to describe  $\delta_2$ . On the other hand, we know from Lemma 3.4 that  $\delta_1(d_j^h) = \bar{\delta}(d_j^h) = a_j^h$  is always an even integer. In view of Lemma 3.6 this means that  $\delta_2(d_j^h) = \delta_2(d_{j+p}^h)$  for all  $j$ . Thus we have (see Figure 3)

$$\begin{aligned} \tilde{u} - 1 - 1 &\equiv -(h-s) - (\tilde{u} - t - (h-s)) \pmod{2p}, \\ -(h-s) + p &\equiv \tilde{u} - 1 \pmod{2p}. \end{aligned}$$

Substituting  $p$  by  $2m-1$  and  $h$  by  $3m-1$  we calculate that  $\tilde{u} = t/2 + 1$  and  $t = 2(s-m)$ . Since  $2 \leq t \leq s$  by Lemma 3.2 we get  $2 \leq 2(s-m) \leq s$  which is equivalent to  $m+1 \leq s \leq 2m$ .  $\square$

Hence the existence of a proper degeneration to the indecomposable  $N$  corresponding to the vertex  $c_1^s$  implies that  $m+1 \leq s \leq 2m$ . Furthermore, if  $M \leq_{\text{deg}} N$  is a proper degeneration to  $N$ , then  $\gamma((\bar{M}, \bar{N})) \in (\bigoplus_{\bar{Y}: Y \text{ indec}} \mathbb{Z}\bar{Y})^*$  is the unique solution of the linear system (14)–(16) with  $c_{u-1}^{s-t+1} = c_{s-m}^{2m-s+1}$ , by Lemma 3.4 and Lemma 3.8. Hence there exists up to isomorphism at most one module  $M$  degenerating to  $N$ .

On the other hand, let  $m + 1 \leq s \leq 2m$  and  $N$  correspond to the vertex  $c_1^s$ . We set  $s' = s$ ,  $t' = 2(s - m)$  and  $u' = t'/2 + 1$  and define  $\delta_2$  according to (17), (18) and (19). Since  $u' - t' \leq 1$  the function  $\delta_2$  is described by Lemma 3.5. In consequence we have  $\delta_2(d_j^h) = \delta_2(d_{j+p}^h)$ . We define  $\delta$  by

$$\delta(c_j^i) = \begin{cases} \delta_2(d_j^i) + \delta_2(d_{j+p}^i) & \text{if } 1 \leq i \leq h - 1 \\ 1/2(\delta_2(d_j^h) + \delta_2(d_{j+p}^h)) & \text{if } h \leq i \leq h + 1. \end{cases}$$

Then  $\delta$  takes value in  $\mathbb{N}$  and solves the linear system (14)–(16). Indeed (14) is a consequence of (17), (15) of (18) and (19), and (16) can be checked easily using Lemma 3.5. Thus  $\delta \in \gamma(\mathcal{S}'_N)$  which means in view of Lemma 2.3 that there exists a proper degeneration to  $N$ . This completes the proof of Theorem 1.1.

Note that the function  $\delta$  constructed above for the indecomposable module  $N$  corresponding to the vertex  $c_1^s$  describes the module  $M$  degenerating to  $N$  in the following way. Let  $P_1, \dots, P_m$  be representatives of the isomorphism classes of projective indecomposable  $\Lambda$ -modules. By Lemma 3.4 and (5)

$$M \cong M_1 \oplus \bigoplus_{i=1}^m P_i^{\delta(\text{rad } P_i)}$$

where  $M_1$  corresponds to the vertex  $c_{s-m+1}^{2m-s}$  if  $s < 2m$  and  $M_1 = 0$  if  $s = 2m$ .

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